PHYSICAL REGIONS OF AN EQUATION OF STATE

George A. Baker, Jr. and J. D. Johnson

Theoretical Division, Los Alamos National Laboratory
University of California, Los Alamos, N. M. 87545, USA

ABSTRACT

There are several key physical regions in any equation of state. We are concerned with astrophysical and similar such applications. The key regions are: (i) All the electrons have coalesced about the nuclei to form atoms (cold curve). Included here would be the further condensation into fluids and solids. (ii) There is a plasma of electrons and nuclei (hot curve). (iii) The low-density, partial-ionization region (ionization curve). The density and temperature go to zero together for this region, because for fixed, non-zero temperature, total ionization occurs in the low density limit. (iv) The intermediate region. Regions (i) and (ii) merge at high density. It is our thesis that region (iv) can be obtained from the hot curve, the ionization curve and the cold curve plus the low order perturbations about them by a suitable interpolation proceedure. By the "cold curve" we must also understand the pendant phase-diagram structure adjoining it. As in our previous work, we will use the Thomas-Fermi equation of state as a test for these ideas. We have previously developed the hot curve, the cold curve and their perturbations. Herein, we discuss the ionization curve. The virtue of our approach is that, if successful, the Thomas-Fermi approximations for these quantities (which have a great many well known defects) can be replaced by the more accurate results of quantum many-body theory to give a greatly improved equations of state.

1. INTRODUCTION AND SUMMARY

The system that we are interested in here is an electrically neutral system of electrons and a single species of nuclei with charge Z. The Coulomb potential energy, as is well known, is neither bounded from below (electron-nuclei part) nor from above (electron-electron part, for example). In addition, it is long-ranged so there is no such thing as a dilute system. Of course, Debye screening does occur, but this feature is a result of the theory, and can not be used *ab initio*. There are two important physical aspects to this problem. They are the quantum mechanical aspect, embodied by Planck's constant, h, and the Coulomb interaction, embodied by the electronic charge, e.

We know that if we take $h = 0$ and $e \neq 0$, then the electrons spiral into the nucleus with the emission of γ-rays, and the whole system collapses into a state of negatively infinite energy. Put otherwise, there is no classical limit for this system, and quantum mechanics is an absolutely essential feature. On the other hand, if we take $h \neq 0$ and $e = 0$, then we get a perfectly well defined system. This system is the ideal Fermi gas for the electrons. In the energy ranges we will be considering, the heavier, less numerous nuclei will behave pretty much like an ideal gas, however a better treatment should not be a particular problem if required. The many-body perturbation theory in e^2 about the $e^2 = 0$ system is, by the nature of the Coulomb potential, a singular one but not[1] a hopeless one.

In the second section, we examine the various limiting physical regions of our system and use this knowledge to construct a physical picture. We will conclude that the best (for our purposes at least) independent variables to use are not the usual temperature and density, but the de Broglie density and the density.

In the third section we discuss Thomas-Fermi theory as a model of the true system and recast it in terms of our independent variables. We review what is both known and useful in our work in this framework. We discuss the dilute-limit ionization curve for this model.

In the final section we show how the information about the various limiting physical regions can be combined in a simple functional form to give a good overall representation. Better information for the various physical regions than that given by the Thomas-Fermi or the Thomas-Fermi-Dirac theories should now be obtainable from many-body perturbation theory, from Saha theory and from various methods for the zero temperature pressure-density curve.

2. PHYSICAL REGIONS

The first physical region is that of the ideal Fermi gas, where $e \equiv 0$. As we will see later, it is best thought of for our purposes as the high-density region. The equation of state is given as,[2,3]

$$\begin{aligned} \frac{P\Omega}{ZNkT} &= g(\zeta), \\ \zeta &= \frac{ZN}{2\Omega}\left(\frac{h^2}{2\pi mkT}\right)^{\frac{3}{2}}, \\ \zeta = f_{\frac{3}{2}}(z), &\quad g(\zeta) = \frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}, \end{aligned} \tag{2.1}$$

where

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{zy^{n-1}e^{-y}\,dy}{1+ze^{-y}}. \tag{2.2}$$

In (2.1), z is determined as a parameter from ζ which in turn is determined from the temperature and density. Then the function $g(\zeta)$ is determined parametrically, and thus the equation of state. Here we use P for the electron pressure, Ω for the volume, m for the mass of the electron, Z for the nuclear charge, N for the number of atoms, T for the temperature, and k for the Boltzmann constant. As remarked above, in this region, the pressure due the the nuclei can just be added to that in (2.1) to give the total pressure.

By means of many-body perturbation theory, we can expand the pressure as,

$$\frac{P\Omega}{ZNkT} = \sum_{j=0} g_j(\zeta)y^j, \tag{2.3}$$

where

$$y = \left[\frac{Ze^2}{r_b kT}\right]^{\frac{1}{2}} = \left(\frac{\sqrt{\pi}}{4}\zeta\right)^{\frac{1}{3}} \sqrt{x}, \tag{2.4}$$

with the definitions

$$\frac{4\pi}{3} r_b^3 = \frac{\Omega}{N}, \quad x = \left(\frac{128Z}{9\pi^2}\right)^{\frac{1}{3}} \frac{me^2 r_b}{\hbar^2}. \tag{2.5}$$

The known terms of (2.3) are $g_0 = g$ of (2.1), $g_1 = 0$, g_2 is the exchange term and g_3 is the Debye-Hückel term. Note that the expansion parameter is proportional to e and not to e^2 because of the singular nature of the perturbation. This feature will not be reproduced in the Thomas-Fermi model theory to the extent it is known.

The other thing we know is that for $T = 0$, thepressure P is a function of the density alone. We expect, in order for this conclusion to hold in a straightforward manner for the $g_j(\zeta)$ of (2.3), that $g_j(\zeta) \propto \zeta^{(2-j)}$ as $\zeta \to \infty$. This feature does hold in Thomas-Fermi theory. Let us recast (2.5) as,

$$P = \left(\tfrac{2}{3}\right)^{\frac{5}{3}} \frac{128Z^{\frac{10}{3}} m^5 e^{10}}{\pi^3 \hbar^8 x^5} \sum_{j=0} \left(\frac{\sqrt{\pi}}{4}\right)^{\frac{j}{3}} \zeta^{(2-j)/3} g_j(\zeta) x^{\frac{j}{2}}, \tag{2.6}$$

where account has been taken of (2.4). It is to be observed, since the density goes to infinity as $x \to 0$, that (2.6) is now a high-density expansion of the pressure at fixed de Broglie density ζ. If we next define

$$\hat{g}_j = \lim_{\zeta \to \infty} g_j(\zeta) \zeta^{(j-2)/3}, \tag{2.7}$$

then we can take the zero temperature (infinite ζ) limit of (2.6), which is

$$P = \left(\tfrac{2}{3}\right)^{\frac{5}{3}} \frac{128Z^{\frac{10}{3}} m^5 e^{10}}{\pi^3 \hbar^8 x^5} \sum_{j=0} \left(\frac{\sqrt{\pi}}{4}\right)^{\frac{j}{3}} \hat{g}_j x^{\frac{j}{2}}. \tag{2.8}$$

This form of viewing the results of Thomas-Fermi theory suggest to us that if we can supply a low-density expansion for fixed ζ, then by use of two-point ($x = 0$ and $x = \infty$) approximation methods for each ζ, we should be able to give a good global representation for the equation of state. The low-density behavior corresponds to the knowledge of the dilute limit ionization curve, which may perhaps be approached through Saha theory.[4]

We now have the following physical picture. First at high ζ, we have the "cold curve" which is a function of density (or x) alone. At high densities this behavior merges into that of the ideal Fermi fluid system and here we have the many-body perturbation theory expansion. Since the $g_j(\zeta)$ are of order unity for small ζ, it is natural to expect that this expansion in $\sqrt{x}$ should have a region of validity of the order of $1/\sqrt[3]{\zeta}$. Thus, in accord with other ways of approaching the same question, we expect that the limiting behavior as $\zeta \to 0$ for fixed density (or x) will be a totally ionized system. Finally, for fixed ζ as $x \to \infty$ the behavior is governed by the dilute-limit, partial-ionization behavior of the system. The question we will be considering in the last section is, can we use our knowledge of the physics of these boundary regions to fill in, to decent accuracy, the equation of state over the whole region?

3. THOMAS-FERMI THEORY

In this section we illustrate the ideas of the previous section on the Thomas-Fermi[5–7] equation of state as a model theory. This theory, and more frequently its generalization the Thomas-Fermi-Dirac theory, is currently quite often used in practical applications, such as the astrophysics of stellar interiors, to give the equation of state over wide ranges of temperature and density.

In the finite-temperature, Thomas-Fermi, statistical theory of the atom, the electron density is given by

$$\rho(r) = \int_0^\infty \frac{2 \cdot 4\pi p^2\, dp/h^3}{\exp[\{p^2/(2m) - eV(r)\}/kT + \eta] + 1}, \tag{3.1}$$

where $-eV(r)$ is the potential energy. Let us define,

$$I_n(\eta) = \int_0^\infty \frac{y^n\, dy}{e^{y-\eta} + 1}. \tag{3.2}$$

Then we can use Poisson's equation to determine $V(r)$,

$$\frac{1}{r}\frac{d^2}{dr^2}[rV(r)] = \frac{16\pi^2}{h^3} e(2mkT)^{\frac{3}{2}} I_{\frac{1}{2}}\left(\frac{eV(r)}{kT} - \eta\right). \tag{3.3}$$

Note that if the electronic charge is taken as $e = 0$, the right-hand side of (3.3) vanishes and the solution for $V(r)$ is just $a + b/r$ where a, b are constants. This solution corresponds to the ideal Fermi gas as described in the previous section.

The usual reduction of these equations to dimensionless variables[5] uses,

$$c = \left(\frac{h^3}{32\pi^2 e^2 m (2mkT)^{\frac{1}{2}}}\right)^{\frac{1}{2}} \propto T^{-\frac{1}{4}}. \tag{3.4}$$

Let $s = r/c$. Then (3.3) becomes,

$$\frac{d^2\beta}{ds^2} = sI_{\frac{1}{2}}\left(\frac{\beta}{s}\right), \tag{3.5}$$

where

$$\frac{\beta}{s} = \frac{eV(r)}{kT} - \eta, \tag{3.6}$$

since η is independent of r. As $r \to 0$, we must have $V(r) \to Ze/r$ to reproduce the nuclear charge. Therefore as one of the boundary conditions for (3.6) we must have,

$$\beta(0) = \alpha = \frac{Ze^2}{kTc} \propto T^{-\frac{3}{4}}. \tag{3.7}$$

In order to insure that the total number of electrons is Z for a neutral atom inside a sphere whose volume is the average volume per atom, Ω/N, a little manipulation[6] shows that,

$$\frac{d\beta}{ds} = \frac{\beta}{s} \text{ at } s = b, \tag{3.8}$$

where $r_b = cb$, see (2.5).

It is convenient for our purposes to re-express the Thomas-Fermi equations in terms of our chosen independent variables, ζ (2.1) and x (2.5).To this end let us make the further change of variables

$$s = \sigma b, \quad \beta = \gamma \alpha. \tag{3.9}$$

The defining equations (3.6-8) now become,

$$\begin{gathered}
\frac{d^2\gamma}{d\sigma^2} = \frac{6\sigma}{\zeta\sqrt{\pi}} I_{\frac{1}{2}}\left(\left(\frac{\sqrt{\pi}}{4}\zeta\right)^{\frac{2}{3}} \frac{x\gamma}{\sigma}\right), \\
\gamma(0) = 1.0, \\
\frac{d\gamma}{d\sigma} = \frac{\gamma}{\sigma} \text{ at } \sigma = 1.0.
\end{gathered} \tag{3.10}$$

From our earlier analysis, one of the regions we need to study is the low density region, $x \to \infty$. To facilitate this study, we note that $I_{\frac{1}{2}}(\eta)$ is a continuously differentiable function that is asymptotically proportional to $\eta^{3/2}$ as $\eta \to +\infty$ and is asymptotically proportional to e^{η} as $\eta \to -\infty$. Thus, in the limit $x \to \infty$ (3.10) becomes,

$$\begin{gathered}
\frac{d^2\gamma}{d\sigma^2} = \begin{cases} \sigma^{-\frac{1}{2}}(x\gamma)^{\frac{3}{2}}, & \gamma > 0, \\ \frac{3\sigma}{\zeta} \exp\left(\left(\frac{\sqrt{\pi}}{4}\zeta\right)^{\frac{2}{3}} \frac{x\gamma}{\sigma}\right), & \gamma < 0, \end{cases} \\
\gamma(0) = 1.0, \\
\frac{d\gamma}{d\sigma} = \frac{\gamma}{\sigma} \text{ at } \sigma = 1.0.
\end{gathered} \tag{3.11}$$

We can observe from (3.11) that when $\gamma < 0$ its second derivative goes exponentially to zero, and so in this region γ is a straight line in σ. The division of the regions between $\gamma > 0$ and $\gamma < 0$ correspond to an ion and the Fermi gas of electrons, respectively. From (3.11), γ starts positive at $\sigma = 0$. The region over which it remains positive can be worked out approximately analytically, or studied numerically. It turns out to tend to zero width as $x \to \infty$. It contains $N < Z$ electrons, and the outer region contains $Z_i = Z - N$ ionized electrons. A little manipulation, similar to that which lead to (3.8), shows that,

$$Z_i = -Z\sigma_0\gamma'(\sigma_0), \tag{3.12}$$

where σ_0 is defined by $\gamma(\sigma_0) = 0$. As remarked above, and as we confirm here, when $\zeta \to 0$ $(T \to \infty)$ we get the total ionization region $Z_i \to Z$. When $\zeta \to \infty$ $(T \to 0)$, $Z_i \to 0$, we get the region where all the electrons condense to form neutral atoms.

Let us now look more closely at the low-temperature region. First, for Thomas-Fermi theory, the pressure is given by,

$$\frac{P\Omega}{N} = \tfrac{2}{9} ZkT \left[\frac{r_b^3}{c^3\alpha}\right] I_{\frac{3}{2}}\left(\left(\frac{\sqrt{\pi}}{4}\zeta\right)^{\frac{2}{3}} x\gamma(1)\right), \tag{3.13}$$

which, in and near the low-temperature limit gives,

$$P = \frac{Z^2e^2}{10\pi\mu^4}\left[\frac{\phi_b}{x}\right]^{\frac{5}{2}}\left[1 + \left(\tfrac{3}{2}\right)^{\frac{4}{3}} \frac{5\pi^2x^2}{12\phi^2\alpha^{\frac{8}{3}}} + O\left(\frac{x^4}{\phi^4\alpha^{\frac{16}{3}}}\right) + O\left(\exp\left[-\frac{E\phi\alpha^{\frac{4}{3}}}{x}\right]\right)\right] \tag{3.14}$$

with E a numerical constant. The defining equations here are

$$\frac{d^2\phi}{d\xi^2} = \xi^{-\frac{1}{2}}\phi^{\frac{3}{2}}, \quad \phi(0) = 1.0, \quad \frac{d\phi}{d\xi} = \frac{\phi}{\xi} \text{ at } \xi = x. \tag{3.15}$$

Note is taken that since by (3.7), $\alpha^{4/3} \propto 1/T$, the last error term in (3.14) contributes no power series terms in T to the expansion about $T = 0$. For the large x (dilute) limit, Baker and Johnson[7] find that

$$\phi(x) \asymp \frac{287.40}{x^3}. \tag{3.16}$$

Thus in the dilute limit,

$$P \propto x^{-10}\left[1 + A(Tx^4)^2 + B(Tx^4)^4 + \cdots + O\left(e^{-C/(Tx^4)}\right)\right], \tag{3.17}$$

or in terms of ζ and x, we get,

$$P \propto x^{-10}\left[1 + A'(\zeta^{-\frac{2}{3}}x^2)^2 + B'(\zeta^{-\frac{2}{3}}x^2)^4 + \cdots + O\left(e^{-C'\zeta^{2/3}/x^2}\right)\right], \tag{3.18}$$

where A, A', B, B', C and C' are numerical constants. As the series is a function of $\zeta^{-\frac{2}{3}}x^2$ alone, in the dilute limit, the simplest hypothesis is that it takes the form

$$P \propto x^{-10} f\left(\zeta^{-\frac{2}{3}}x^2\right) \underset{x\to\infty}{\longrightarrow} Kx^{-10}\left(\frac{\sqrt{x}}{\zeta^{\frac{1}{6}}}\right)^{\gamma}, \tag{3.19}$$

where K and γ are constants to be determined. Since we have failed to find them analytically, we have fit them to the numerical solutions of the Thomas-Fermi equations as solved for aluminum at a compression of 10^{-6} for a range to temperature from 0.25 – 20 ev. We find that

$$\gamma \approx 7.0 \pm 0.2. \tag{3.20}$$

This result gives the general dilute limit asymptote for the Thomas-Fermi pressure. Needless to say, the other error term will modify the situation significantly away from this limit, and is expected to introduce a ζ-dependence in even the first term of the expansion off the large x limit.

4. GLOBAL REPRESENTATION OF THE THOMAS-FERMI PRESSURE

Since our goal is to create a "zipper" formula into which the separately computed or measured boundary-physical-region behaviors can be simply "zipped," and since the cold curve and its pendant phase diagram is far too complex for general treatment, we write,

$$P = \Delta P + P_{\text{cold curve}}. \tag{4.1}$$

Baker and Johnson[7] have previously computed for the Thomas-Fermi model the expansion of P about the "hot curve," or the high-density expansion as we saw in (2.6). It is,

$$P = 196.6889 Z^{\frac{10}{3}} x^{-5}\zeta^{-\frac{2}{3}}\left[g_0(\zeta) + \left(\frac{\sqrt{\pi}}{4}\zeta\right)^{\frac{2}{3}} g_2(\zeta)x + + \left(\frac{\sqrt{\pi}}{4}\zeta\right)^{\frac{4}{3}} g_2(\zeta)x^2 + \cdots\right], \tag{4.2}$$

where P is in megabars, the terms $g_1(\zeta) = g_3(\zeta) = 0$, and Baker and Johnson[7] give the representations, accurate to about 0.1%,

$$
\begin{aligned}
g_0(\zeta) &\approx \left[\frac{1 + 0.61094880\zeta + 0.12660436\zeta^2 + 0.0091177644\zeta^3}{1 + 0.080618739\zeta}\right]^{\frac{1}{3}}, \\
g_2(\zeta) &= -\frac{3}{10}, \\
g_4(\zeta) &\approx \frac{97}{2880}\left[\frac{v_3(\zeta)}{u_5(\zeta)}\right]^{\frac{1}{3}}.
\end{aligned}
\tag{4.3}
$$

The definitions,

$$
\begin{aligned}
v_3(\zeta) =& 1 + 0.17549205\zeta + 1.1833437 \times 10^{-2} + 3.0923597 \times 10^{-4}\zeta^3, \\
u_5(\zeta) =& 1 + 1.2361522\zeta + 0.54327035\zeta^2 + 9.7985998 \times 10^{-2}\zeta^3, \\
& + 6.1912639 \times 10^{-3}\zeta^4 + 1.6191557 \times 10^{-4}\zeta^5,
\end{aligned}
\tag{4.4}
$$

have been used in (4.3). When we subtract the cold curve to form ΔP, we must subtract the large-ζ limiting behavior from the g_j's. When this step is done, we get

$$\Delta P = 196.6889 Z^{\frac{10}{3}} x^{-5}\left[\tilde{g}_0(\zeta) + \tilde{g}_4(\zeta)x^2 + \cdots\right], \tag{4.5}$$

where

$$
\begin{aligned}
\tilde{g}_0(\zeta) =& g_0(\zeta)\zeta^{-\frac{2}{3}} - 0.48359758, \\
\tilde{g}_2(\zeta) \equiv& 0, \\
\tilde{g}_4(\zeta) =& 0.33782096 g_4(\zeta)\zeta^{\frac{2}{3}} - 0.041787498.
\end{aligned}
\tag{4.6}
$$

Since in the dilute limit, by (3.20),

$$\Delta P \propto x^{-\frac{13}{2}}, \tag{4.7}$$

then the [ ] in (4.5) should sum to something proportional to $x^{-3/2}$. After some experimentation, we have chosen the form,

$$\Delta P \approx \frac{196.6889 Z^{\frac{10}{3}} x^{-5}\tilde{g}_0(\zeta)}{\left[1 - 2\frac{\tilde{g}_4(\zeta)}{\tilde{g}_0(\zeta)}x^2 + \left\{\frac{Z^{\frac{4}{3}}\zeta^{\frac{7}{6}}\tilde{g}_0(\zeta)}{d_0}\right\}^2 x^3\right]^{\frac{1}{2}}}. \tag{4.8}$$

This form agrees with the series expansion in x, and if the asymptotic form (3.19) for $x \to \infty$ were exact, then d_0 would be a constant. Because of the necessary corrections to that asymptotic form we have chosen the approximate representation,

$$d_0 \approx \frac{1602.7288 + (19532.475\zeta^{\frac{1}{6}} - 12736.679)\zeta^{\frac{1}{6}}/\sqrt{x}}{1 + 35.483743\zeta^{\frac{1}{6}}/\sqrt{x}}. \tag{4.9}$$

This form corresponds to the asymptotic result (3.19) and two corrections to it. The form (4.9) is derivable as the [1/1] Padé approximant to series,

$$
\begin{aligned}
\Delta P \asymp 196.6889 Z^2 x^{-\frac{13}{2}}\zeta^{-\frac{7}{6}}[&1602.7288 + (19532.475\zeta^{\frac{1}{6}} - 69607.496)\zeta^{\frac{1}{6}}/\sqrt{x} \\
&+ (-693085.33\zeta^{\frac{1}{6}} + 2469934.6)\zeta^{\frac{1}{3}}/x + \cdots],
\end{aligned}
\tag{4.10}
$$

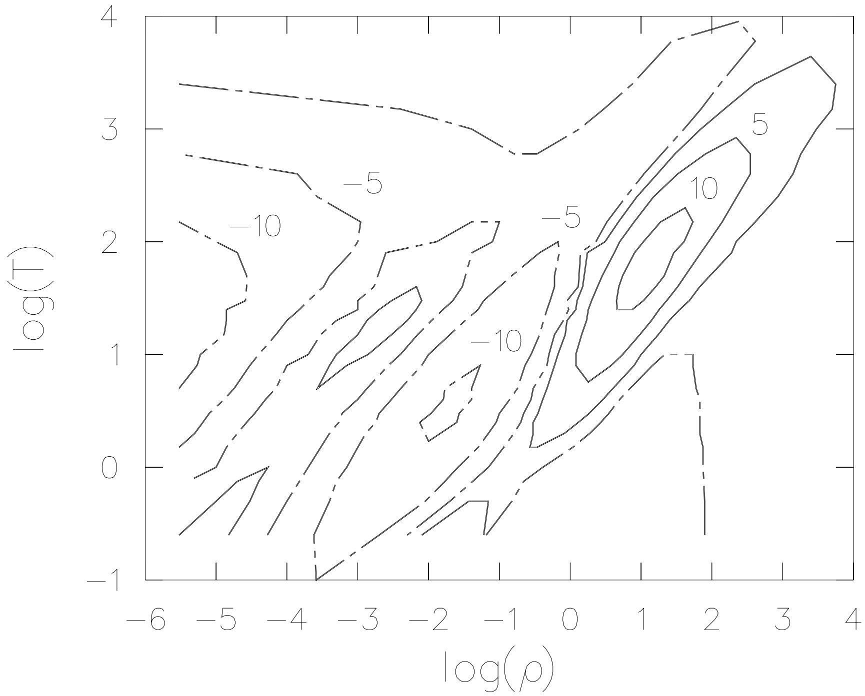

Figure 1. Error contours for the representation (4.1, 8, 11) of the Thomas-Fermi pressure. Positive errors, the fit larger than Thomas-Fermi, are solid lines while negative are chain-dashed. The 5 and 10 percent contours are explicitly labeled while the ± 1 percent ones have no labels. Here the axes are the log, base 10, of the temperature in electron volts and the density in g/cc for aluminum. The conversion from ev to Kelvin is 11604.

which was derived by the expansion of (4.9). Form (4.9) is a rough but adequate representation of d_0 for our needs. Since by identification with (3.19), d_0 should be positive and since we are trying to keep our results simple, we found it to be computationally more efficient to simply cut-off d_0 rather than to generate a more elaborate form which doesn't need a cutoff. We use the maximum of the result of (4.9) and 50 for d_0. This number was choosen to be just a little lower than the smallest, well-determined value of d_0 found by a scan of direct Thomas-Fermi theory computation through our region of application. This value occurs at very high temperature and density.

In addition we have used Baker and Johnson's[7] representation for the cold curve pressure. It is based on the use of the $T = 0$ part of (3.14) and is,

$$P_{\texttt{cold curve}} \approx 9.0549692 \left(\frac{Z\rho}{\mathcal{G}}\right)^{\frac{5}{3}} x^{\frac{5}{2}} \left[\frac{(1 + 1.59659x^{-0.772} + 1.06595x^{-1.544})^{\frac{1}{6}}}{1 + 0.2783436x^{-0.772}}\right]^{9.715} \tag{4.11}$$

where $P_{\texttt{cold curve}}$ is in megabars, the density ρ is in grams per cc, and $\mathcal{G}$ is the gram molecular weight.

We conclude by showing in Fig. 1 a contour map for the percent errors of our fit to Thomas-Fermi. The maximum deviation is about 12% and occurs in two islands along the "seam" between the high and low density regions. The error also rises for very low density, but we are not as interested in that. The rms error is about 4%. The representation is doing very well especially when we consider the large ranges over which we are working, about nine decades in density and almost five in temperature. We hope in future work to obtain representations of the internal energy and free energy. We also wish to include in our representation better physics than

Thomas-Fermi.

One of the authors (G.B.) wishes to thank the USARO for partially supporting his attendance at the *XVI International Workshop on Condensed-Matter Theories*, where this paper was presented.

REFERENCES

1. D. J. Thouless. "The Quantum Mechanics of Many-Body Systems," Academic Press, New York (1961).
2. K. Huang. "Statistical Mechanics," John Wiley and Sons, New York (1963).
3. G. A. Baker, Jr. and J. D. Johnson, Thomas-Fermi equation of state-the hot curve, *in*: "Condensed Matter Theories, Volume 5," V. C. Aquilera-Navarro, ed., Plenum Press, New York (1990).
4. M. N. Saha, Article title, *Phil. Mag.* 40:472 (1920); D. Mihalas, "Stellar Atmospheres," W. H. Freeman & Co., San Francisco (1970).
5. R. P. Feynman, N. Metropolis and E. Teller, Equations of state of elements based on the generalized Fermi-Thomas theory, *Phys. Rev.* 75:1561 (1949).
6. R. D. Cowan and J. Ashkin, Extension of the Thomas-Fermi-Dirac statistical theory of the atom to finite temperatures, *Phys. Rev.* 105:144 (1957).
7. G. A. Baker, Jr. and J. D. Johnson, General structure of the Thomas-Fermi equation of state, *Phys. Rev. A* 44:2271 (1991).